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New non-skew symmetric classical r-matrices and 'twisted' quasigraded Lie algebras

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Abstract

We construct a family of quasigraded Lie algebras that admit the Kostant–Adler scheme. They coincide with quasigraded deformations of the loop algebras in different gradings. Using them we explicitly construct new non-skew-symmetric classical r-matrices with spectral parameters. We consider examples of the constructed algebras and constructed r-matrices corresponding to the homogeneous quasigrading, Z_2-quasigrading and to the principle quasigrading in detail.

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1. Introduction

One of the most important objects in the theory of classical integrable systems is the classical r-matrix. The classical (non-dynamical) r-matrix is a $\mathfrak{g}\otimes\mathfrak{g}$-valued function (here $\mathfrak{g}$ is semisimple or reductive classical matrix Lie algebra) of two complex variables λ and μ that satisfy the 'generalized' classical Yang–Baxter equation [1–3]:

$$[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu)] = [r_{23}(\mu,\nu), r_{12}(\lambda,\mu)] - [r_{32}(\nu,\mu), r_{13}(\lambda,\nu)], \qquad (1)$$

where $r_{12}(\lambda,\mu) \equiv r(\lambda,\mu)\otimes 1$ etc.

The classical r-matrix gives a possibility of defining the Poisson brackets between the matrix elements of some Lax matrix $L(\lambda)$ belonging to the space of $\mathfrak{g}$-valued functions of λ:

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)] - [r_{21}(\mu,\lambda), L_2(\mu)]. \qquad (2)$$

Bracket (2) possesses a lot of commuting functions that can be obtained as a decomposition in the powers of λ of the generating functions $\mathrm{Tr}\, L^k(\lambda)$ [2, 3]. Hence, each solution of equation (2) gives the possibility of defining *classical integrable* Hamiltonian systems with the Hamiltonian being one of the above commuting functions.

Usually the main attention in the theory of non-dynamical classical r-matrices and associated integrable systems is devoted to the so-called skew-symmetric r-matrices [4–10] such that $r_{21}(\mu, \lambda) = -r_{12}(\lambda, \mu)$. This is explained by the fact that using the skew-symmetric r-matrix one can construct not only *classical integrable systems* but also *quantum integrable systems* that are known as *Gaudin spin chains*.

As was shown in our previous papers [15, 16] with each non-skew symmetric solution of equation (1), it is also possible to associate *quantum integrable systems* that generalize usual Gaudin spin chains and coincide with the quantization of the systems of classical integrable interacting tops proposed by us in [17]. This fact makes the problem of the construction of non-skew symmetric solutions of equation (1) very important both from the point of view of classical and quantum integrable systems.

In the present paper, we construct new non-skew symmetric r-matrices $r_{12}(\lambda, \mu)$. In order to find new non-skew-symmetric solutions of the generalized classical Yang–Baxter equation we use a connection of this equation with the theory of the infinite-dimensional Lie algebras [11]. We rely on the fact that each infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$-valued functions of λ that admits a decomposition into a direct sum of two subalgebras $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$ (Kostant–Adler scheme) also admits the classical r-matrix coinciding with the kernel of the operator R [11, 12], where $R = \frac{1}{2}(P_+ - P_-)$ and $P_\pm$ are the projection operators on the subalgebras $\widetilde{\mathfrak{g}}_\pm$.

Hence, in order to construct new classical r-matrices it is necessary to construct Lie algebras $\tilde{\mathfrak{g}}$ admitting decomposition into a direct sum of two subalgebras. In our previous papers [18–22], we have proposed for this role special quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A$ with the decomposition $\widetilde{\mathfrak{g}}_A = \widetilde{\mathfrak{g}}_{A+} + \widetilde{\mathfrak{g}}_{A-}$ realized as the loop space $\mathfrak{g}(\lambda^{-1}, \lambda)$ with a new Lie bracket, deformed with the help of some matrix A. In the papers [25–28], we combined ideas [18–22] and ideas of [23, 24] and defined new types of the quasigraded Lie algebras admitting Kostant–Adler scheme. They coincide with the 'twisted' subalgebras of the Lie algebras $\widetilde{\mathfrak{g}}_A$ defined with the help of a Z_p-grading of finite-dimensional Lie algebras $\mathfrak{g}$, or, equivalently, with some its automorphism σ of the order p. It turned out that for the special choice of the matrices A it is possible to define twisted subalgebras $\widetilde{\mathfrak{g}}_A^\sigma \subset \widetilde{\mathfrak{g}}_A$ in the completely analogous way as for the case of ordinary loop algebras $L(\mathfrak{g})$ [24, 29]. The choice of the corresponding matrix A depends on the Z_p-grading of $\mathfrak{g}$ or, equivalently, on its automorphism σ of order p. In the present paper, we establish simple criteria for such the matrices A to define Lie algebra $\widetilde{\mathfrak{g}}_A^\sigma$. We consider in detail cases of the 'homogeneous' gradation, associated with the trivial automorphism σ, Z_2 gradation associated with an involutive automorphism σ and principal gradation associated with the Coxeter automorphism and explicitly find out the form of the corresponding matrices A.

In order to construct classical non-skew symmetric r-matrices using the Lie algebra $\widetilde{\mathfrak{g}}_A^\sigma$ we construct two realizations of $\widetilde{\mathfrak{g}}_A^\sigma$ in the spaces of matrix-valued functions of λ with the usual undeformed commutator. The first one is a realization in the space of $\mathfrak{g}$-valued irrational functions of λ and the second one is the 'rational' realization of $\widetilde{\mathfrak{g}}_A^\sigma$ as a special quasigraded subalgebra of $gl(n)$ loop algebra. In the first realization we obtain for the corresponding r-matrices the following explicit formulae:

$$r_A^\sigma(\lambda, \mu) = \frac{\sum_{j=0}^{p-1} \lambda^j \mu^{p-j} \sum_{\alpha=1}^{\dim \mathfrak{g}_{\bar{j}}} A(\lambda)^{1/2} X_\alpha^{\bar{j}} A(\lambda)^{1/2} \otimes A(\mu)^{-1/2} X^{\overline{-j},\alpha} A(\mu)^{-1/2}}{(\lambda^p - \mu^p)}, \tag{3}$$

where $A(\lambda) = 1 - A\lambda$, $X_\alpha^{\bar{j}}$ is the basis element of the graded subspace $\mathfrak{g}_{\bar{j}}$, $X^{\overline{-j},\alpha}$ is the dual basic element of the subspace $\mathfrak{g}_{\overline{-j}}$. In the second realization we obtain the 'gauge-equivalent' r-matrix $r_A^{\sigma'}(\lambda, \mu)$ (see formula (26)). In the loop algebra limit $A \to 0$ both formulae give us standard r-matrix of Avan and Talon [30] of the loop algebra in the Z_p grading.

We consider in detail examples of r-matrices associated with the homogeneous grading, Z_2 grading and principal grading. We show that in the homogeneous case ($p = 1$) we recover the 'anisotropic' r-matrix constructed in our previous paper [17].

The structure of the present paper is the following: in the second section we construct quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A$ and their twisted subalgebras, in the third section we construct their realizations, and at last in the fourth section we obtain the corresponding new classical r-matrices and consider different examples.

2. Quasigraded Lie algebras with K–A decomposition

2.1. 'Homogeneous' quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A$

Let us remind the following definition [32]:

Definition 2.1. *Infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ is called $\mathbb{Z}$-quasigraded of type (p, q) if it admits the following decomposition:*

$$\widetilde{\mathfrak{g}} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j, \qquad \text{such that} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \sum_{k=-p}^{q} \mathfrak{g}_{i+j+k}.$$

The following proposition holds true [20].

Proposition 2.1. *Let $\widetilde{\mathfrak{g}}$ be $\mathbb{Z}$-quasi graded of type $(0, 1)$, or $(1, 0)$. Then $\widetilde{\mathfrak{g}}$ admits a decomposition into the sum of its two subalgebras $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$.*

In order to construct $\mathbb{Z}$-quasigraded algebras of type (0, 1) we will deform a Lie algebraic structure in loop algebras. We will introduce the new Lie bracket into $L(\mathfrak{g}) = \mathfrak{g}\otimes \operatorname{Pol}(\lambda, \lambda^{-1})$:

$$[X \otimes p(\lambda), Y \otimes q(\lambda)]_F = [X, Y] \otimes p(\lambda)q(\lambda) - F(X, Y) \otimes \lambda p(\lambda)q(\lambda), \tag{4}$$

where $X, Y \in \mathfrak{g}$, $p(\lambda), q(\lambda) \in \operatorname{Pol}(\lambda, \lambda^{-1})$, $[\ ,\ ]$ on the righthand side of this identity denotes the ordinary Lie bracket in $\mathfrak{g}$ and the map $F : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is skew. It is evident by the very construction that the Lie algebras with the bracket (4) are the $\mathbb{Z}$-quasi graded algebras of type (0, 1) with the quasigrading being defined in the standard way by degrees of the spectral parameter λ.

By the direct calculation one can prove the following proposition [20].

Proposition 2.2. *For bracket (4) to satisfy Jacobi identities the cochain F should satisfy the following two requirements:*

$(J1)\ \sum_{c.p.\{i,j,k\}} (F([X_i, X_j], X_k) + [F(X_i, X_j), X_k]) = 0,$

$(J2)\ \sum_{c.p.\{i,j,k\}} F(F(X_i, X_j), X_k) = 0.$

Let $\mathfrak{g}$ be a classical matrix Lie algebra of the type $gl(n), so(n)$ and $sp(n)$ over the field of the complex or real numbers. We will realize algebra $so(n)$ as follows: $so(n) = \{X \in gl(n) \mid X = -sX^\top s\}$, where s is some constant symmetric matrix such that $s^2 = 1$, and the algebra $sp(n)$ is defined as follows: $sp(n) = \{X \in gl(n)|X = sX^\top s\}$, where n is an even number, s is a constant skew-symmetric matrix such that $s^2 = -1$.

As it follows from the results of [33] the following proposition holds true.

Proposition 2.3. *Let $\mathfrak{g}$ be a classical matrix Lie algebra over the field $\mathbb{K}$ of complex or real numbers. Let us define the numerical ($\mathbb{K}$-valued) $n \times n$ matrix A of the following type:*

(1) A is arbitrary for $\mathfrak{g} = gl(n)$,
(2) $A = sA^{\top}s$ for $\mathfrak{g} = so(n)$,
(3) $A = -sA^{\top}s$ for $\mathfrak{g} = sp(n)$.

Then the maps $F_A : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ of the form $F_A(X, Y) = [X, Y]_A = XAY - YAX$ are the correctly defined skew symmetric maps that satisfy conditions (J1)–(J2).

We will denote the infinite-dimensional Lie algebra with the Lie bracket given by (4) by $\widetilde{\mathfrak{g}}_A$ and the finite-dimensional Lie algebra $\mathfrak{g}$ with the bracket $[,]_A$ by $\mathfrak{g}_A$.

The Lie bracket in the algebra $\widetilde{\mathfrak{g}}_A$ has the following explicit form:

$$[X(\lambda), Y(\lambda)]_{F_A} = [X(\lambda), Y(\lambda)]_{A(\lambda)} \equiv [X(\lambda), Y(\lambda)] - \lambda[X(\lambda), Y(\lambda)]_A, \tag{5}$$

where $X(\lambda), Y(\lambda) \in L(\mathfrak{g}) = \mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$, $A(\lambda) \equiv 1 - \lambda A$.

2.2. 'Twisted' quasigraded Lie algebras $\widetilde{\mathfrak{g}}^{\sigma}_A$

In this subsection, we will define another class of the quasigraded Lie algebras of type (0, 1). They will coincide with the 'twisted' subalgebras of the algebras $\widetilde{\mathfrak{g}}_F$.

Let $\mathfrak{g} = \sum_{k=0}^{p-1} \mathfrak{g}_{\bar{k}}$ be a $\mathbb{Z}/p\mathbb{Z}$ grading of $\mathfrak{g}$. Let us consider the following subspace in $\widetilde{\mathfrak{g}}_F$:

$$\widetilde{\mathfrak{g}}^{\sigma}_F = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\bar{j}} \otimes \lambda^j, \tag{6}$$

where $\bar{j}$ denotes the class of equivalence of the elements $j \in \mathbb{Z} \bmod p\mathbb{Z}$.

The next proposition holds true.

Proposition 2.4. *Subspace $\widetilde{\mathfrak{g}}^{\sigma}_F$ is the closed Lie subalgebra in $\widetilde{\mathfrak{g}}_F$ if and only if*

$$F(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) \subset \mathfrak{g}_{\overline{i+j+1}}. \tag{7}$$

Remark 1. It is known that the $\mathbb{Z}/p\mathbb{Z}$ grading of $\mathfrak{g}$ may be defined with the help of some automorphism σ of the order p. If we extend automorphism σ to the map $\hat{\sigma}$ of the whole algebra $\widetilde{\mathfrak{g}}_F$, defining its action on the space $\mathfrak{g} \otimes Pol(\lambda, \lambda^{-1})$ in the standard way [29]: $\hat{\sigma}(X \otimes \lambda^k) = \sigma(X) \otimes e^{-2\pi i k/p}\lambda^k$, then the subalgebra $\widetilde{\mathfrak{g}}^{\sigma}_F$ can be defined as a set of its stable points:

$$\widetilde{\mathfrak{g}}^{\sigma}_F = \{X \otimes p(\lambda) \in \widetilde{\mathfrak{g}}_F \mid \sigma(X \otimes p(\lambda)) = X \otimes p(\lambda)\}.$$

From the very definition of $\widetilde{\mathfrak{g}}^{\sigma}_F$ and the commutation relation in $\widetilde{\mathfrak{g}}_F$ the next result follows.

Proposition 2.5. *For the cocycles F satisfiying (7) the algebra $\widetilde{\mathfrak{g}}^{\sigma}_F$ is Z-quasigraded of type* (0, 1).

From this proposition follows, in particular, that the algebra $\widetilde{\mathfrak{g}}^{\sigma}_F$ admits the direct sum decomposition $\widetilde{\mathfrak{g}}^{\sigma}_F = \widetilde{\mathfrak{g}}^{\sigma+}_F + \widetilde{\mathfrak{g}}^{\sigma-}_F$, where

$$\widetilde{\mathfrak{g}}^{\sigma+}_F = \bigoplus_{j \geqslant 0} \mathfrak{g}_{\bar{j}} \otimes \lambda^j, \qquad \widetilde{\mathfrak{g}}^{\sigma-}_F = \bigoplus_{j < 0} \mathfrak{g}_{\bar{j}} \otimes \lambda^j. \tag{8}$$

Remark 2. It is easy to see that the Lie algebra $\widetilde{\mathfrak{g}}_F$ itself may be viewed as the special case of the subalgebras $\widetilde{\mathfrak{g}}^{\sigma}_F$ corresponding to the $\sigma \equiv id$.

Now let us pass to the case of the matrix Lie algebras and cochains F_A given by proposition (2.3). In this case, condition (7) can be written in more detail.

Indeed, let us define the map $\mathcal{A} : \mathfrak{g} \to \mathfrak{g}$ by the following formula:

$$\mathcal{A}(X) = 1/2(AX + XA).$$

Then the following proposition holds true.

Proposition 2.6. *Let $\mathfrak{g}$ be a classical matrix Lie algebra and the cochain F has the form described in the proposition 2.3. Then condition (7) is satisfied if and only if*

$$\mathcal{A}(\mathfrak{g}_{\bar{i}}) \subset \mathfrak{g}_{\overline{i+1}}. \tag{9}$$

Proof. It is easy to show that the cochain F_A can be rewritten in the following form:

$$F_A(X, Y) = [\mathcal{A}(X), Y] + [X, \mathcal{A}(Y)] - \mathcal{A}([X, Y]).$$

From this it immediately follows that condition (7) is satisfied if and only if $\mathcal{A}(\mathfrak{g}_{\bar{i}}) \subset \mathfrak{g}_{\overline{i+1}}$. □

That proves the proposition.

It is also possible to rewrite condition (9) in the other form.

Proposition 2.7. *Conditions (7) and (9) are satisfied if and only if*

$$\sigma \circ \mathcal{A} = e^{2\pi\,\mathrm{i}/p} \mathcal{A} \circ \sigma. \tag{10}$$

Proof. By the very definition of the automorphism σ that corresponds to the chosen Z_p grading of $\mathfrak{g}$ we have that $\sigma(\mathfrak{g}_{\bar{k}}) = e^{2\pi\,\mathrm{i}k/p}\mathfrak{g}_{\bar{k}}$ i.e. on the subspaces $\mathfrak{g}_{\bar{k}}$ the automorphism σ acts by multiplication by $e^{2\pi\,\mathrm{i}k/p}$. Taking into account that $\mathcal{A}(\mathfrak{g}_{\bar{k}}) \subset \mathfrak{g}_{\overline{k+1}}$, the linearity of $\mathcal{A}$ and the fact that $\mathfrak{g} = \sum_{k=0}^{p-1} \mathfrak{g}_{\bar{k}}$ we obtain the statement of the proposition. □

In some cases this condition may be rewritten in a more explicit way as a condition on matrix A itself. By the direct verification one can prove the following corollary.

Corollary 2.1. *Let an automorphism σ of $\mathfrak{g} \subset gl(n)$ could be lifted to the automorphism of $gl(n)$ as an associative algebra. Then condition (9) is satisfied if and only if*

$$\sigma(A) = e^{2\pi\,\mathrm{i}/p} A.$$

Remark 3. The condition of this corollary is satisfied, for example, in the case when the automorphism σ is internal, i.e. $\sigma(X) = w_0 X w_0^{-1}$ for some $w_0 \in G$.

We will denote Lie algebra $\widetilde{\mathfrak{g}}^{\sigma}_F$ defined with the help of the cocycle F_A by $\widetilde{\mathfrak{g}}^{\sigma}_A$.

2.2.1. Case of involutive automorphism ($\sigma^2 = \mathrm{id}$). Let us consider the case of the automorphism of the second order $\sigma^2 = id$, corresponding Lie algebras $\widetilde{\mathfrak{g}}^{\sigma}_A$. In this case we have the following decomposition: $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}} = \mathfrak{k} + \mathfrak{p}$. In this case algebra $\widetilde{\mathfrak{g}}^{\sigma}_A \subset \widetilde{gl(n)}_A$ has the following decomposition:

$$\widetilde{\mathfrak{g}}^{\sigma}_A = \sum_{j\in Z} \mathfrak{k}^{(2j)}\lambda^{2j} + \mathfrak{p}^{(2j+1)}\lambda^{2j+1}.$$

Commutation relations in the algebra $\widetilde{gl(n)}^{\sigma}_A$ have the form

$$[\mathfrak{k}^{(2i)}, \mathfrak{k}^{(2j)}]_F \subset \mathfrak{k}^{(2(i+j)} + \mathfrak{p}^{(2(i+j)+1)}, [\mathfrak{k}^{(2i)}, \mathfrak{p}^{(2j+1)}]_F \subset \mathfrak{p}^{(2(i+j)+1)} + \mathfrak{k}^{(2(i+j)+2)},$$
$$[\mathfrak{p}^{(2i+1)}, \mathfrak{p}^{(2j+1)}]_F \subset \mathfrak{k}^{(2(i+j)+2)} + \mathfrak{p}^{(2(i+j+1)+1)}.$$

The form of matrix A that satisfies condition (9), strongly depends on the explicit form of the automorphism σ. There are two substantially different situations.

Proposition 2.8. *(i) If an automorphism σ of $\mathfrak{g} \subset gl(n)$ is lifted to an automorphism of algebra $gl(n)$ as an associative algebra then condition (7) is satisfied if and only if*

$$\sigma(A) = -A. \tag{11}$$

(ii) If an automorphism σ of $\mathfrak{g} \subset gl(n)$ is lifted to a minus anti-automorphism of algebra $gl(n)$ as an associative algebra then condition (7) is satisfied if and only if

$$\sigma(A) = A. \tag{12}$$

Proof. Item (i) of this proposition follows from corollary 2.1. Item (ii) of the proposition is checked by the direct verification using the explicit form of the cocycle F_A. □

Remark 4. In the case $\mathfrak{g} = gl(n)$ conditions (11) and (12) are equivalent to the condition $A \in \mathfrak{g}_1$ and $A \in \mathfrak{g}_0$ respectively.

Let us consider the following examples.

Example 1. Let $\mathfrak{g} = gl(n)$ and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = gl(p) + gl(q)$, $\mathfrak{g}_{\bar{1}} \simeq \mathbb{C}^{2pq}$, $p + q = n$. The subspaces $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ have the following explicit form:

$$\mathfrak{g}_{\bar{1}} = \mathfrak{p} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \qquad \mathfrak{k} = \mathfrak{g}_{\bar{0}} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

$D_1 \in \mathrm{Mat}(p)$, $D_2 \in \mathrm{Mat}(q)$, $B \in \mathrm{Mat}(p, q)$, $C \in \mathrm{Mat}(q, p)$.

It is easy to show that the corresponding automorphism σ is given by the formula: $\sigma(X) = wXw^{-1}$ where $w = \mathrm{diag}(1_p, -1_q)$, and, hence, is lifted to an automorphism of $gl(n)$ viewed as an associative algebra. Hence, as it follows from the arguments above: $A \in \mathfrak{g}_{\bar{1}}$ or, equivalently $A = \left(\begin{smallmatrix} 0 & A_1 \\ A_2 & 0 \end{smallmatrix}\right)$.

Example 2. This example may be viewed as a restriction of example 1. Let $\mathfrak{g} = so(n)$ in the realization by skew-symmetric matrices for which $s \equiv \mathbf{1}_n$, and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = so(p) + so(q)$, $\mathfrak{g}_{\bar{1}} \simeq \mathbb{C}^{pq}$, $p + q = n$. The subspaces $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ have the following explicit form:

$$\mathfrak{g}_{\bar{0}} = \mathfrak{k} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \qquad \mathfrak{g}_{\bar{1}} = \mathfrak{p} = \begin{pmatrix} 0 & -C^t \\ C & 0 \end{pmatrix},$$

where $S_1 \in so(p)$, $S_2 \in so(q)$. Then, as in the previous case this grading corresponds to the automorphism σ defined by the formula: $\sigma(X) = wXw^{-1}$ where $w = \mathrm{diag}(1_p, -1_q)$. Hence, matrix A should satisfy the condition $\sigma(A) = -A$ and, in addition, be symmetric. In the result we obtain that $A = \left(\begin{smallmatrix} 0 & A_1 \\ A_1^t & 0 \end{smallmatrix}\right)$.

Example 3. Let $\mathfrak{g} = gl(n)$ and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = so(n)$, $\mathfrak{g}_{\bar{1}} \simeq \mathrm{Symm}(n)$. It is easy to see that the corresponding automorphism σ is given by the formula: $\sigma(X) = -sX^{\top}s$ (where s is symmetric matrix such that $s^2 = 1$) and, hence, is lifted to the minus antiautomorphism of $gl(n)$ viewed as an associative algebra. Hence, as it follows from the arguments above, $\sigma(A) = A$, or, equivalently, $A \in so(n)$.

Example 4. Let $\mathfrak{g} = gl(n)$, n is even number, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ and $\mathfrak{g}_{\bar{0}} = sp(n)$. It is easy to see that the corresponding automorphism σ is given by the formula: $\sigma(X) = sX^{\top}s$, where s is a matrix of standard symplectic structure in the space $\mathbb{C}^n$ ($s^2 = -1$). It is lifted to the minus antiautomorphism of $gl(n)$ viewed as an associative algebra. Hence, as it follows from the arguments above, $\sigma(A) = A$, or, equivalently, $A \in sp(n)$.

2.2.2. Case of Coxeter automorphism and 'principal' quasigrading. Let an algebra $\mathfrak{g}$ with the bracket [,] be semisimple (reductive) classical Lie algebra of the rank r. Let $\mathfrak{h} \subset \mathfrak{g}$ be its Cartan subalgebra, $\Delta_{\pm}$ be its set of positive(negative) roots, Π the set of simple roots, $H_i \in \mathfrak{h}$ is the basis of Cartan subalgebra $E_\alpha, \alpha \in \Delta$ is the corresponding root vectors.

Let us define the so-called principal grading of $\mathfrak{g}$ [29], putting

$$\deg H_i = 0, \quad \deg E_{\alpha_i} = 1, \quad \deg E_{-\alpha_i} = -1.$$

It is evident that in such a way we obtain the grading of $\mathfrak{g}$: $\mathfrak{g} = \sum_{k=0}^{h-1} \mathfrak{g}_{\bar{k}}$ with the graded subspaces $\mathfrak{g}_{\bar{k}}$ be defined as follows: $\mathfrak{g}_{\bar{k}} = \mathrm{Span}_C\{E_\alpha\}$, where α is the root of the height k i.e. $\alpha = \sum_{i=1}^r k_i\alpha_i$ if $\alpha \in \Delta_+, \alpha = -\sum_{i=1}^r k_i\alpha_i$ if $\alpha \in \Delta_-$ and $k = \sum_{i=1}^r k_i, h$ is a Coxeter number of $\mathfrak{g}$. In particular, $\mathfrak{g}_{\bar{0}} = \mathfrak{h}$, $\mathfrak{g}_{\bar{1}} = \mathrm{Span}_C\{E_{\alpha_i}, E_{-\theta}|\alpha_i \in \Pi\}$, $\mathfrak{g}_{\overline{-1}} = \mathrm{Span}_C\{E_{-\alpha_i}, E_\theta|\alpha_i \in \Pi\}$ and θ is the highest root of the height $h-1$.

If the cocycle F satisfies condition (7) we may define the corresponding 'principal' quasigraded Lie algebra $\widetilde{\mathfrak{g}}^\sigma_A$ in the following way:

$$\widetilde{\mathfrak{g}}^\sigma_A = \sum_{m\in\mathbb{Z}} \sum_{j\in 0}^{h-1} \mathfrak{g}^{(m)}_{\bar{j}} \otimes \lambda^{j+mh}, \qquad \text{where} \quad \mathfrak{g}^{(m)}_{\bar{j}} \simeq \mathfrak{g}_{\bar{j}}. \tag{13}$$

Let us pass to the consideration of classical matrix Lie algebras. It will be convenient to chose the realizations of the algebras $so(n)$ and $sp(n)$ such that the Cartan subalgebra will coincide with the algebra of diagonal matrices. For this purpose we will use the following matrices s in our definition of the algebras $so(n)$ and $sp(n)$: $s = \mathrm{diag}(1, s_{2n})$, where $s_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$, if $\mathfrak{g} = so(2n+1)$; $s \equiv s_{2n}$ if $\mathfrak{g} = so(2n)$; $s = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ if$\mathfrak{g} = sp(n)$. Now we can find matrices A that satisfy condition (9) explicitly. The following proposition holds true [26].

Proposition 9. *The cochain F_A satisfies condition (7) if and only if the matrix A has the form*

1. $A = \sum_{i=1}^{n-1} a_i X_{ii+1} + a_n X_{n1}$ *if* $\mathfrak{g} = gl(n)$
2. $A = \sum_{i=1}^{n-1} a_i (X_{i+1,i+2} + X_{n+i+2,n+i+1}) + a_n(X_{1+n,1} + X_{1,2n+1}) + a_{n+1}(X_{2+n3} + X_{3+n2})$ *if* $\mathfrak{g} = so(2n+1)$
3. $A = \sum_{i=1}^{n-1} a_i (X_{i,i+1} + X_{n+i+1,n+i})$ *if* $\mathfrak{g} = sp(n)$
4. $A = \sum_{i=1}^{n-1} a_i (X_{i,i+1} + X_{n+i+1,n+i}) + a_n(X_{n,2n-1} + X_{n-1,2n}) + a_{n+1}(X_{1+n2} + X_{2+n1})$ *if* $\mathfrak{g} = so(2n)$.

3. Two realizations of the Lie algebra $\widetilde{\mathfrak{g}}^\sigma_A$

In this section, we will describe two realizations of the Lie algebras $\widetilde{\mathfrak{g}}^\sigma_A$ in the spaces of rational and irrational functions of λ with and ordinary non-deformed Lie brackets. Following remark 2 we will hereafter consider the Lie algebra $\widetilde{\mathfrak{g}}_A$ to be a special case of the Lie algebras $\widetilde{\mathfrak{g}}^\sigma_A$ corresponding to the trivial automorphism σ.

3.1. Realization of $\widetilde{\mathfrak{g}}^\sigma_A$ and $\widetilde{\mathfrak{g}}^{\sigma*}_A$ in the space of irrational functions

In this subsection, we will describe a realization of the algebra $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}^\sigma_A$ in the space of $\mathfrak{g}$-valued irrational functions on λ. By the direct verification one can easily prove the following proposition.

Proposition 3.1. *There exists a homomorphism $\phi^{(1)}_A$ from the algebra $\widetilde{\mathfrak{g}}^\sigma_A$ into the algebra of $\mathfrak{g}$-valued functions on λ equipped with the standard Lie bracket: $[X(\lambda), Y(\lambda)] = X(\lambda)Y(\lambda) - Y(\lambda)X(\lambda)$ given by the formula: $\phi^{(1)}_A(X(\lambda)) \equiv A(\lambda)^{1/2}X(\lambda)A(\lambda)^{1/2}$.*

Remark 5. It is easy to see that $\ker\phi_A^{(1)} = 0$ and, hence this homomorphism provides us an exact realization of the $\widetilde{\mathfrak{g}}_A^{\sigma}$ in a space of $\mathfrak{g}$-valued irrational functions.

Let $X_\alpha^{\bar{j}}$ be the basis element of the graded subspace $\mathfrak{g}_{\bar{j}}$. Then the basis in the algebra $\widetilde{\mathfrak{g}}_A^{\sigma}$ in this realization consists of the following matrix-valued functions:

$$\widetilde{X_\alpha^{\bar{j}}} = \lambda^j A(\lambda)^{1/2} X_\alpha^{\bar{j}} A(\lambda)^{1/2}, \qquad \text{where} \quad j \in \mathbb{Z}. \tag{14}$$

Let us introduce the standard pairing between $\widetilde{\mathfrak{g}}_A^{\sigma *}$ and $\widetilde{\mathfrak{g}}_A^{\sigma}$:

$$\langle X, Y\rangle = \mathrm{res}_{\lambda=0}\lambda^{-1}\,\mathrm{Tr}(X(\lambda)Y(\lambda)), \tag{15}$$

where $X(\lambda) \in \widetilde{\mathfrak{g}}_A^{\sigma}, Y(\lambda) \in \widetilde{\mathfrak{g}}_A^{\sigma *}$. Taking into account (see [29]) that $(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) = 0$ if $i + j \neq 0 \bmod p$ we conclude that the basis of the dual space $\widetilde{\mathfrak{g}}_A^{\sigma *}$ with respect to the natural pairing (15) consists of the following functions:

$$\widetilde{Y_\alpha^{\bar{j}}} = \lambda^{-j} A(\lambda)^{-1/2} X^{\overline{-j},\alpha} A(\lambda)^{-1/2}, \tag{16}$$

where $X^{\overline{-j},\alpha}$ is an element dual to $X_\alpha^{\bar{j}}$.

Example 5. Let us consider the case of the 'homogeneous quasigrading'($\sigma \equiv id$) and the diagonal matrix A: $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. In this case the algebras $\widetilde{\mathfrak{g}}_A$ have a natural interpretation as Lie algebras of a special meromorphic functions on higher genus curves. Indeed, introducing the following notations: $\lambda_i = (\lambda^{-1} - a_i)^{1/2}$ it is easy to see that λ_i satisfy the second-order algebraic equations:

$$\lambda_i^2 - \lambda_j^2 = a_j - a_i, \qquad i, j = 1, n. \tag{17}$$

Equations (17) define embedding of the algebraic curve $\mathcal{H}$ in the linear space $\mathbb{C}^n$ with the coordinates $\lambda_1, \ldots, \lambda_n$. The genus of this curve grows with the growth of n. It covers the standard hyperelliptic curve defined by the following equation: $y^2 = \prod_{i=1}^n (\lambda^{-1} - a_i)$.

The basis in the spaces $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}_A^*$ consists of the following matrix-valued functions on the curve $\mathcal{H}$:

$$\widetilde{X_{ij}^m} = \lambda^{m+1}\lambda_i\lambda_j X_{ij} \qquad \text{and} \qquad \widetilde{Y_{ij}^m} = \lambda^{-m-1}\lambda_i^{-1}\lambda_j^{-1} X_{ji}, \quad i, j = 1, n, m \in \mathbb{Z},$$

where X_{ij} is a basis of a matrix Lie algebra $\mathfrak{g}$. For example, for the case $\mathfrak{g} = gl(n)$ we have that $X_{ij} = I_{ij}$, where $(I_{ij})_{ab} = \delta_{ai}\delta_{bj}$ for the case $\mathfrak{g} = so(n)$ (in the realization by skew-symmetric matrices for which $s \equiv \mathbf{1}_n$) we have that $X_{ij} = I_{ij} - I_{ji}$ etc.

3.2. Realization of $\widetilde{\mathfrak{g}}_A^{\sigma}$ and $\widetilde{\mathfrak{g}}_A^{\sigma *}$ as a subalgebras of $L(gl(n))$

In this subsection, we will describe a realization of the algebra $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}_A^{\sigma}$ in the space of $gl(n)$-valued rational functions of λ.

The following proposition holds true.

Proposition 3.2. *There exists a homomorphism $\phi_A^{(2)}$ from the algebras $\widetilde{\mathfrak{g}}_A^{\sigma}$ into the algebra of $gl(n)$-valued rational functions of λ equipped with the standard Lie bracket: $[X(\lambda), Y(\lambda)] = X(\lambda)Y(\lambda) - Y(\lambda)X(\lambda)$ defined by the following formula: $\phi_A^{(2)}(X(\lambda)) \equiv A(\lambda)X(\lambda)$.*

Proof. It follows from the following formal equality $\phi_A^{(2)} = \mathrm{Ad}_{A(\lambda)^{1/2}} \cdot \phi_A^{(1)}$, where $A(\lambda)^{1/2}$ may be viewed as an element of the formal Lie group $Gl(n)((\lambda))$. □

Remark 6. It is easy to see that $\ker\phi_A^{(2)} = 0$ and, hence this homomorphism provides us the exact realization of the $\widetilde{\mathfrak{g}}_A^{\sigma}$ in a space of $gl(n)$-valued rational functions of λ i.e. as the subalgebras of the loop algebra $L(gl(n))$. It is necessary to emphasize that contrary to the graded subalgebras of the loop algebras, constructed quasigraded subalgebras are not isomorphic to the corresponding loop algebras. Although there is a homomorphizm $\phi_A^{(2)}$ of $\widetilde{\mathfrak{g}}_A^{\sigma}$ into $L(gl(n))$ it could not be interpreted as an isomorphism, because it has no correctly defined inverse:map $(\phi_A^{(2)})^{-1}$ contains the formal power series $(1 + A\lambda + A^2\lambda^2 + \cdots)$, image of which does not belong to the space $\widetilde{\mathfrak{g}}_A^{\sigma}$.

Let $X_\alpha^{\bar{j}}$ be the basis element of the graded subspace $\mathfrak{g}_{\bar{j}}$. Then the basis in the algebra $\widetilde{\mathfrak{g}}_A^{\sigma}$ in this realization consists of the following matrix-valued functions:

$$\widetilde{X_\alpha^{\bar{j}}} = \lambda^j A(\lambda) X_\alpha^{\bar{j}} = \lambda^j X_\alpha^{\bar{j}} - \lambda^{j+1} A X_\alpha^{\bar{j}}, \qquad \text{where} \qquad j \in \mathbb{Z}. \tag{18}$$

Using the standard pairing (15) between $\widetilde{\mathfrak{g}}_A^{\sigma *}$ and $\widetilde{\mathfrak{g}}_A^{\sigma}$ and taking into account (see [29]) that $(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) = 0$ if $i + j \neq 0 \bmod p$ we obtain that the basis of the dual space $\widetilde{\mathfrak{g}}_A^{\sigma *}$ consists of the following functions:

$$\widetilde{Y_\alpha^{\bar{j}}} = \lambda^{-j} X^{\overline{-j},\alpha} A^{-1}(\lambda). \tag{19}$$

4. Classical r-matrices with spectral parameter and infinite-dimensional algebras with K–A decomposition

4.1. General construction

In this subsection, we will remind the notion of a classical R-operator, its connection with solutions of equation (1) and infinite-dimensional Lie algebras (see [1, 11–13]). All the facts from this subsection will be essentially used in the next section for obtaining new solutions of (1).

Let $\widetilde{\mathfrak{g}}$ be hereafter some infinite-dimensional Lie algebra of $\mathfrak{g}$-valued functions of one complex variable λ with the natural Lie bracket [,]. Let $\widetilde{\mathfrak{g}}^*$ be the dual space to $\widetilde{\mathfrak{g}}$ with respect to the natural pairing:

$$\langle X(\lambda), L(\lambda)\rangle = \mathrm{res}_{\lambda=0}\lambda^{-1}(X(\lambda), L(\lambda)), \tag{20}$$

where $X(\lambda) \in \widetilde{\mathfrak{g}}$, $L(\lambda) \in \widetilde{\mathfrak{g}}^*$ and (,) is some invariant form on $\mathfrak{g}$. Let the Lie algebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$-valued functions of λ admit linear space decomposition into the direct sum of two subalgebras: $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$. It is known [11] that in such a case the operator:

$$R = 1/2(P_+ - P_-), \tag{21}$$

where P_+, P_- are projection operators onto the subalgebras $\widetilde{\mathfrak{g}}_\pm$, satisfies the modified Yang–Baxter equation [11, 13] and defines the so-called R-matrix bracket on $\widetilde{\mathfrak{g}}$.

It is also known (see [12]) that if R-operator possesses the kernel:

$$R(X)(\lambda) = \oint_{\mu=0} (r_{12}(\lambda, \mu), X_2(\mu))_2 \, \mathrm{d}\mu, \tag{22}$$

where $r_{12}(\lambda, \mu)$ is a $\mathfrak{g} \otimes \mathfrak{g}$-valued function of two complex variables, $X_2 \equiv 1 \otimes X$, (,) is an invariant non-degenerated bilinear form on $\mathfrak{g}$, then the function $r_{12}(\lambda, \mu)$ satisfies the 'generalized' Yang–Baxter equation:

$$[r_{12}(\lambda, \mu), r_{13}(\lambda, \nu)] = [r_{23}(\mu, \nu), r_{12}(\lambda, \mu)] - [r_{32}(\nu, \mu), r_{13}(\lambda, \nu)], \tag{23}$$

where $r_{12}(\lambda, \mu) \equiv r_{1,2}(\lambda, \mu) \otimes 1$ etc.

Remark 7. Note that equation (23) has an additional symmetry in comparison with a standard classical Yang–Baxter equation, namely it is invariant with respect to the transformation $r_{12}(\lambda,\mu) \to f(\mu) r_{12}(\lambda,\mu)$ where $f(\mu)$ is an arbitrary function.

Combining the above facts we obtain the following proposition.

Proposition 4.1. *Let a Lie algebra* $\widetilde{\mathfrak{g}}$ *of* $\mathfrak{g}$*-valued functions of* λ *admit a linear space decomposition into the direct sum of two subalgebras:* $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$*. Let* $\widetilde{X_\alpha^m} \equiv \widetilde{X_\alpha^m}(\lambda)$*, where* $m \in Z, \alpha \in 1, \dim\mathfrak{g}$ *be the basis in* $\widetilde{\mathfrak{g}}$ *agreed with this decomposition i.e.* $\widetilde{X_\alpha^m} \in \widetilde{\mathfrak{g}}_+$ *form* $\geqslant 0$, $\widetilde{X_\alpha^m} \in \widetilde{\mathfrak{g}}_-$ *if* $m < 0$ *and* $\alpha \in 1, \dim\mathfrak{g}$*. Let* $\widetilde{Y_\alpha^m} \equiv Y_\alpha^m(\mu)$ *be the basis in the linear space* $\widetilde{\mathfrak{g}}^*$ *dual to the basis* $\widetilde{X_\alpha^m}$ *in* $\widetilde{\mathfrak{g}}^*$ *with respect to introduced above pairing* $\langle , \rangle$ *(20).*

Then the function $r_{12}(\lambda,\mu)$ *of the form*

$$r_{1,2}(\lambda,\mu) = 1/2\left(\sum_{\alpha=1}^{\dim\mathfrak{g}}\sum_{m\geqslant 0}\widetilde{X_\alpha^m}(\lambda)\otimes\widetilde{Y_\alpha^m}(\mu) - \sum_{\alpha=1}^{\dim\mathfrak{g}}\sum_{m<0}\widetilde{X_\alpha^m}(\lambda)\otimes\widetilde{Y_\alpha^m}(\mu)\right) \tag{24}$$

satisfies the generalized classical Yang–Baxter equation (23).

In the previous section, we have constructed a large family of Lie algebras that possess the decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$. Using them in the next subsection we will explicitly construct the corresponding classical r-matrices.

4.2. New non-skew r-matrices from Lie algebras $\widetilde{\mathfrak{g}}_A^\sigma$

Let us now construct a family of the classical r-matrices that correspond to the Lie algebras $\widetilde{\mathfrak{g}}_A^\sigma$. Using results of subsections 4 and 3.1 we obtain the following statement.

Theorem 4.1. *Let* $\mathfrak{g}$ *be one of the classical matrix Lie algebras* $gl(n), so(n)$ *or* $sp(n)$ *and* σ *be their automorphism of order* p *that defines* Z_p *grading of* $\mathfrak{g}$*. Let a matrix* A *satisfies conditions of proposition 2.3 and condition (10). Then the following* $\mathfrak{g}\otimes\mathfrak{g}$*-valued irrational function of the two complex variables* λ *and* μ*:*

$$r_A^\sigma(\lambda,\mu) = \frac{\sum_{j=0}^{p-1}\lambda^j\mu^{p-j}\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{j}}} A(\lambda)^{1/2} X_\alpha^{\bar{j}} A(\lambda)^{1/2}\otimes A(\mu)^{-1/2} X^{\overline{-j},\alpha} A(\mu)^{-1/2}}{(\lambda^p-\mu^p)} \tag{25}$$

satisfies the generalized classical Yang–Baxter equation (23).

Proof. Theorem is proved by an application of formula (24), using the definition of the basis in the Lie algebra $\widetilde{\mathfrak{g}}_A$ and dual space $\widetilde{\mathfrak{g}}_A^*$ (14), (16) where $(X_\alpha^{\bar{j}})^* = X^{\overline{-j},\alpha}$ and the expansion of $1/(\lambda^p-\mu^p)$ in the power series in $(\lambda/\mu)^p$ and $(\mu/\lambda)^p$. □

Remark 8. Using the realization of $\widetilde{\mathfrak{g}}_A^\sigma$ in the space of the $gl(n)$-valued rational functions, described in subsection 3.2, or, equivalently, making a gauge transformation of the classical r-matrix (25), we obtain that it could also be written in the rational form:

$$r_A^{\sigma'}(\lambda,\mu) = \frac{\sum_{j=0}^{p-1}\lambda^j\mu^{p-j} A_1(\lambda)\left(\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{j}}} X_\alpha^{\bar{j}}\otimes X^{\overline{-j},\alpha}\right)A_2^{-1}(\mu)}{(\lambda^p-\mu^p)}, \tag{26}$$

where $A_1(\lambda) = A(\lambda)\otimes 1$, $A_2^{-1}(\mu) = 1\otimes A^{-1}(\mu)$. Let us note that in this case for all classical matrix Lie algebras $\mathfrak{g}$ the r-matrix $r_A^{\sigma'}(\lambda,\mu)$ takes values in $gl(n)\otimes gl(n)$ but not in $\mathfrak{g}\otimes\mathfrak{g}$.

4.2.1. Case of 'homogeneous' grading ($\sigma = id$). Let $\sigma = id$ and matrix A be diagonal. In this case formula (25) acquires a simpler form:

$$r_A(\lambda,\mu) = \frac{\lambda}{(\lambda-\mu)} \sum_{i,j=1}^{n} \frac{\lambda_i\lambda_j}{\mu_i\mu_j} X_{ij}\otimes X_{ji}, \tag{27}$$

where $\lambda_i^2 = (\lambda^{-1} - a_i)$, $\mu_i^2 = (\mu^{-1} - a_i)$.

This r-matrix coincides with the so-called 'anisotropic' r-matrix of [17]. It has a nice geometrical interpretation: it could be viewed as the 'higher genus r-matrix' in the sense that functions λ_i and μ_j are living on the covering of the standard hyperelliptic curve. This r-matrix is equivalent to the non-skew deformation of the rational r-matrix of Yang. It stands for integrability of the generalized tops, generalized Steklov–Liapunov systems, generalized Clebshch and Neumann systems, their 'spin' generalizations [19–21] and new classically integrable Gaudin-type magnets [17].

Let us now consider the small rank and small genus examples of the r-matrix (27).

Example 6. Let us consider the case $\mathfrak{g} = so(3)$, $A = \mathrm{diag}(a_1, a_2, a_3)$ in the formula (27):

$$r_A(\lambda,\mu) = \frac{\lambda}{(\lambda-\mu)} \sum_{i<j}^{3} \frac{\lambda_i\lambda_j}{\mu_i\mu_j} X_{ij}\otimes X_{ij} = \frac{\lambda}{(\lambda-\mu)} \frac{\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3} \sum_{k=1}^{3} \frac{\mu_k}{\lambda_k} X_k\otimes X_k, \tag{28}$$

where $X_k = \epsilon_{ijk}X_{ij}$. Multiplying this expression by $\mu\mu_1\mu_2\mu_3$ (see remark 6) we obtain

$$\begin{aligned} r_A(\lambda,\mu) &= \frac{1}{(\mu^{-1}-\lambda^{-1})} \sum_{k=1}^{3} \frac{\lambda_1\lambda_2\lambda_3}{\lambda_k} \mu_k X_k\otimes X_k \\ &= \sum_{k=1}^{3} (\lambda_k(u+v) - \lambda_k(u-v)) X_k\otimes X_k, \end{aligned} \tag{29}$$

where λ_k, μ_k are expressed via the Jacobi elliptic functions of the uniformizing parameters u, v:

$$\lambda_1 = \frac{1}{sn(u)}, \qquad \lambda_2 = \frac{dn(u)}{sn(u)}, \qquad \lambda_3 = \frac{cn(u)}{sn(u)},$$
$$\mu_1 = \frac{1}{sn(v)}, \qquad \mu_2 = \frac{dn(v)}{sn(v)}, \qquad \mu_3 = \frac{cn(v)}{sn(v)},$$

λ^{-1} and μ^{-1} are the Weierstrass functions of the parameters u and v respectively, and we have used addition formula [34] that permits to express $\lambda_k(u\pm v)$ via $\lambda_k(u) \equiv \lambda_k$, and $\lambda_k(v) \equiv \mu_k$.

4.2.2. Case of Z_2 grading ($\sigma^2 = id$). Let us consider the case of the automorphism of the second-order $\sigma^2 = id$, and the corresponding decomposition: $\mathfrak{g} = \mathfrak{g}_{\bar 0} + \mathfrak{g}_{\bar 1} = \mathfrak{k} + \mathfrak{p}$. In this $\mathfrak{k}^* = \mathfrak{k}$, $\mathfrak{p}^* = \mathfrak{p}$. Let X_α^+ and X_α^- be a basis in the linear spaces $\mathfrak{k}$ and $\mathfrak{p}$ respectively, $X^{+,\alpha}$ and $X^{-,\alpha}$ be the dual basis in these spaces and matrix A satisfies conditions (10). Then expression (26) acquires the form

$$r_A^{\sigma'}(\lambda,\mu) = \frac{1}{(\lambda^2-\mu^2)} A_1(\lambda) \left(\mu^2 \sum_{\alpha=1}^{\dim\mathfrak{k}} X_\alpha^+\otimes X^{+,\alpha} + \lambda\mu \sum_{\alpha=1}^{\dim\mathfrak{p}} X_\alpha^-\otimes X^{-,\alpha} \right) A_2^{-1}(\mu). \tag{30}$$

Example 7. Let $\mathfrak{g} = gl(2)$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Z_2-grading of $gl(2)$ where

$$\mathfrak{k} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad \mathfrak{p} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.$$

A basis if the linear space $\mathfrak{k}$ and $\mathfrak{p}$ constitute matrices

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Matrix A that satisfies conditions (10) has the form:

$$A = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}.$$

Taking into account that in this case $A^{-1}(\mu) = (1 + \mu A)(1 + a_1 a_2 \mu^2)^{-1}$ and multiplying r matrix (30) by $(1 + a_1 a_2 \mu^2)$ we obtain

$$r_A^{\sigma'}(\lambda, \mu) = \frac{1}{(\lambda^2 - \mu^2)}(1 - A\lambda) \otimes 1(\mu^2(H_1 \otimes H_1 + H_2 \otimes H_2) + \lambda\mu(X \otimes Y + Y \otimes X))1 \otimes (1 + A\mu). \tag{31}$$

This r-matrix stands for integrability of the different integrable deformations of the mKdV and sine-Gordon equations ([25]), in particular, Calodgero–Degasperis and modified sine-Gordon equations.

4.2.3. Case of the 'principal' grading ($\sigma^h = id$). Let us consider the case of the 'principal' grading. In this case formula (26) acquires the form

$$r_A^{\sigma_c'}(\lambda, \mu) = \frac{\mu^h}{(\lambda^h - \mu^h)}\left(A_1(\lambda)\left(\sum_{i=1}^{\dim \mathfrak{h}} H_i \otimes H_i + \sum_{\alpha \in \Delta} \lambda^{l(\alpha)} \mu^{-l(\alpha)} E_\alpha \otimes E_{-\alpha}\right) A_2(\mu)^{-1}\right), \tag{32}$$

where matrix A is defined as in proposition 2.9, $l(\alpha)$ is a height of the root α, E_α is a basis vector of the corresponding root space, H_i is a basis vector of the Cartan subalgebra $\mathfrak{h} \equiv \mathfrak{g}_0$ with the following normalization: $(E_\alpha, E_{-\alpha}) = 1, (H_i, H_i) = 1$.

This r-matrix is exactly a deformation of a trigonometric r-matrix standing behind integrability of ordinary Toda chain. The r-matrix $r_A^{\sigma_c}(\lambda, \mu)$ in its turn provides integrability of the 'deformed' Toda chain of [26] and of the modified Toda field equations [27].

Example 8. In the case $\mathfrak{g} = gl(n) r$-matrix (32) could be written as

$$r_A^{\sigma_c'}(\lambda, \mu) = \frac{1}{(\lambda^n - \mu^n)} \sum_{k=0}^{n-1} \lambda^k \mu^{n-k} A_1(\lambda) \left(\sum_{i-j=k \bmod n} X_{ij} \otimes X_{ji} \right) A_2(\mu)^{-1}, \tag{33}$$

where X_{ij} is the standard basis of $gl(n)$: $(X_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Taking into account the explicit form of the matrix $A(\lambda)$ it is easy to calculate that in the case $\mathfrak{g} = gl(n)$

$$A^{-1}(\mu) = (1 + \mu A)(1 + \mu^n \det A)^{-1}.$$

Taking into account that a non-skew symmetric r-matrix is defined up to the multiplication by the arbitrary function of μ (remark 6), and multiplying r-matrix (33) by $(1 + \mu^n \det A)$ we obtain

$$r_A^{\sigma_c'}(\lambda, \mu) = \frac{1}{(\lambda^n - \mu^n)} \sum_{k=0}^{n-1} \lambda^k \mu^{n-k} \left(\sum_{i-j=k \bmod n} (1 - A\lambda) X_{ij} \otimes X_{ji} (1 + A\mu) \right), \tag{34}$$

where $A = \sum_{i=1}^{n-1} a_i X_{ii+1} + a_n X_{n1}$. In the case $a_i = 1$ this r-matrix stands for integrability of the periodic closure of the infinite Voltera coupled system [35].

It is easy to see that in the particular case $n = 2$ formula (34) yields expression (31).

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